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Exercise 4 - Numerical methods for fluid-structure interaction (Winter term 2015)

Exercise 4.1: Show that the fluid incompressibility condition $\nabla \cdot v = 0$ reads in ALE_{fx} coordinates

$$\widehat{\nabla} \cdot (\hat{J}\widehat{F}^{-1}\hat{v}) = 0.$$

The proof can be restricted to $\mathbb{R}^3$. What is the surprising result?
Remark: It also holds $\widehat{\nabla} \cdot (\hat{J}\widehat{F}^{-1}\hat{v}) = \hat{J}\text{trace}(\widehat{\nabla}\hat{v}\widehat{F}^{-1})$.

Exercise 4.2: Transform the Eulerian divergence of stress $-\nabla \cdot \sigma_f$ into the weak ALE_{fx} form. Homogeneous Dirichlet conditions can be assumed on the boundary.
Hint: There are two possible ways for this calculation.

Exercise 4.3: Let a stationary fluid-structure interaction system in $\Omega = \Omega_f \cup \Omega_s$ be given. Let $V := H_0^1$ and $L_0 := L^2/\mathbb{R}$ be function spaces. Then for stationary Navier-Stokes, with a given $f_f \in H^{-1}$, we seek a vector-valued velocity v, a scalar-valued pressure p such that (in strong form):

$$\begin{aligned} v \cdot \nabla v - \nabla \cdot \sigma &= f_f \quad \text{in } \Omega_f, \\ \nabla \cdot v &= 0 \quad \text{in } \Omega_f, \end{aligned}$$

with $\sigma = -pI + \frac{1}{2}(\nabla v + \nabla v^T)$. Furthermore, we seek for given $\hat{f}_s \in H^{-1}$ vector-valued displacements u such that

$$-\hat{\nabla} \cdot (\hat{F}\hat{\Sigma}) = \hat{f}_s \quad \text{in } \Omega_s,$$

where $\hat{F} = \nabla \hat{u} + I$ and $\hat{\Sigma} = 2\mu \hat{E} + \lambda tr(\hat{E})I$, $\hat{E} = (\hat{F}\hat{F}^T - I)$.
Tasks:

- Derive an ALE_{fx} formulation in weak and strong form of the coupled FSI problem (Hint: Write the Navier-Stokes system in the reference configuration);
- What are the coupling conditions between the two systems?

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Exercise 4.4: Consider mass transfer of dissolved chemical solutes. Let D be the (molar) diffusion constant, v a given velocity, and f some given right hand side forces. We seek for a scalar-valued concentration $c \in L^2([0,T], H_0^1)$ with initial condition $c(0) = c_0$ such that (in strong form):

$$\partial_t c + \nabla \cdot (vc - D\nabla c) = f(c) \quad \text{in } \Omega_c. \tag{0.0.1}$$

Tasks:

- Derive the weak formulation while assuming that the boundary is split into a Dirichlet part $\Gamma_D > 0$ and a Neumann $\Gamma_N > 0$ part such that $\partial\Omega = \Gamma_D \cup \Gamma_N$;
- Derive the ALE$_{dm}$ formulation in weak and strong form;
- Derive the ALE$_{fx}$ formulation in weak and strong form;
- What is the stress tensor in the above system and how does this tensor read in ALE$_{fx}$ form?
- Let (0.0.1) hold in the domain Ω_c. Couple a solid domain $\hat{\Omega}_s$ to $\hat{\Omega}_c$ with $\overline{\Omega}_c \cap \overline{\Omega}_s = \hat{\Gamma}_i$. In $\hat{\Omega}_s$, we seek vector-valued displacements $\hat{u}$ to the stationary solid equation:

  $$-\hat{\nabla} \cdot (\hat{F}\hat{\Sigma}) = \hat{f}_s \quad \text{in } \hat{\Omega}_s.$$

  Question: What are the coupling conditions on $\hat{\Gamma}_i$ using the ALE$_{fx}$ approach if we assume that the concentration is zero in $\hat{\Omega}_s$?

Discussion of exercises: Nov 23 / Nov 30, 2015