Betreff: Meeting 14
Von: wick@ifam.uni-hannover.de <wick@ifam.uni-hannover.de>
Datum: 26.01.21, 18:23
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IntNumPDE, Meeting 14 $\boxed{\text{Welcome}}$

Jan 26, 2021
14:15-15:45

Plan for today

5) FEM class in July? ↓ again email to me.

1) Questions (emails) from your side (Thanks a lot!)
↳ 1.1) Differences FD vs. FEM
↳ 1.2) Application of Banach spaces, H^1, H^1_0, L^2 → Meeting 8+9, Chapter 7 in lecture notes

Main point of H^1 etc. in comparison to classical spaces that they allow for more general solution concept.

2) Short questions (Quiz, Chapter 17)

3) All from Peru / South America who need a certificate of participation, please write me an email (Name, Family name, University)

4) → Exam for Peruvian participants
↳ non-official, please write me as well an email

To 1), specifically 1.1):

FD: approximation of derivatives

FEM: Use given equation (PDE, ODE), multiply by test function and integrate

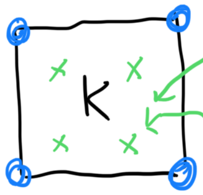
(test function: has same properties as sought unknown solution. Vary "variational" over all these functions to identify solution.)

(+) Advantages FD: very simple to realize, quick implementation, easy to understand (ex. in Chapter 6)

(−) Drawbacks FD: complex geometries, boundary conditions,
↳ for instance Neumann cond. are naturally included in FEM

nodal points

stress value evaluations: $\sigma(u)$, ∇u: better at nodal points



$-\Delta u = f$

$\downarrow$

$\int_K \nabla u \cdot \nabla \varphi \, dx = \int_K f \cdot \varphi \, dx$

$K \hat{=}$ element in $\Omega \subset \mathbb{R}^2$

↳ integrals are usually evaluated with Gaussian numerical quadrature

↳ quadrature points

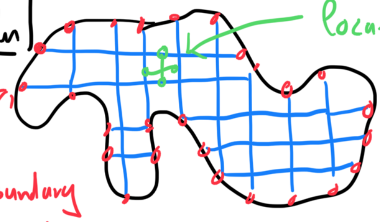
stress value evaluations: $\sigma(u)$, ∇u: better at Gauß points in FEM, rather nodal points in FD,

error analysis needs more regularity than in FEM,

local mesh adaptivity more difficult to realize

$\|u-u_h\| \le \|u^{(4)}\|$ FD

$\|u-u_h\| \le \|u^{(2)}\|$ FEM



Boundary points

local mesh adaptivity (increasing accuracy of solution locally)

(+) FEM: variational framework with an in-depth theory, local mesh adaptivity, error analysis, Galerkin approximations, many codes (open-source, commercial) available, discont. Galerkin formulations (relation to finite volume methods)

(−) FEM: from scratch it takes a bit of time to implement, understanding of weak formulations, need strong mathematical background for error analysis

Quiz (Chapter 17)

(1) A: $-\Delta u = f$ in Ω (PDE)

$u = 0$ on $\partial\Omega$ (BC)

Q: What is here Laplacian?

$\Omega \subset \mathbb{R}^d$, then $\Delta u = \sum_{i=1}^{d} \partial_{ii} u$

Q: Variational Form

square-integrable on Ω

1) Function spaces: $H_0^1(\Omega) = \{ u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega), u = 0 \text{ on } \partial\Omega \}$

homogeneous Dirichlet conditions

2) Multiply by $\varphi \in H_0^1(\Omega)$ and integrate:

test function

$$-\Delta u = f$$

$$\rightarrow -\int_\Omega \Delta u \, \varphi \, dx = \int_\Omega f \cdot \varphi \, dx$$

Int. by parts

$$\rightarrow \int_\Omega \nabla u \cdot \nabla \varphi \, dx - \cancel{\int_{\partial\Omega} \nabla u \cdot n \, \varphi \, ds} = \int_\Omega f \cdot \varphi \, dx$$

$\rightarrow 0$

$\varphi \in H_0^1(\Omega)$

$\rightarrow \varphi = 0$ on $\partial\Omega$

$$\Leftrightarrow \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx$$

$(\cdot,\cdot)$ notation

$$\Leftrightarrow (\nabla u, \nabla \varphi) = (f, \varphi)$$

Q1 from Chapter 17

② Idea of FEM?

conforming FEM

We take finite dimensional subspace $V_h \subset H_0^1$ and divide Ω into smaller subdomains $\mathcal{T}_h$.
On each subdomain, we assign degrees of freedom and construct unique polynomials (basis functions of V_h).
We then evaluate integrals $\int \nabla u \cdot \nabla \varphi$, $\int f \cdot \varphi$ on each $K \in \mathcal{T}_h$. This yields a linear equation system,

local element

which afterward must be solved. The solution represents the coefficients of $u_h \in V_h$, i.e. $u_h = \sum_{i=1}^{n} u_i \varphi_i$

coefficients

Q2

④ FD: s.o.

Ex. $-u''(x) = f$ → $-u''(x) = \dfrac{2u(x) - u(x-h) - u(x+h)}{h^2}$

2nd order central difference quotient.

⑤ see lecture notes

(6) s.o.

(7) Discretization error

A: It is the error measured in an appropriate norm between the (unknown) exact solution u and the discretized solution u_h.

Q: Ex. for FEM?

$$\|u - u_h\|_{L^2} = \sqrt{\left(\int_\Omega (u-u_h)^2 \, dx\right)}$$

$$\leq c\, h^2 \|f\|_{L^2}$$

Q: Order?

A: The convergence order is 2 w.r.t. h.

Q: Is this result optimal?

A: Yes. But as you showed us in the lecture, we would usually obtain

$\|u - u_h\| \leq c\, h \|f\|$ (only order 1!)

But with the Aubin-Nitsche trick, we can get one order more in h!

Q: What is the convergence order in H^1?

A:

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Q: what is the convergence order in H^1?

A: $\|u-u_h\|_{H^1} \le c\, h\, \|u\|_{H^2}$

Q: Now we have two norms L^2 and H^1. Which estimate is better?

A: $\|u-u_h\|_{H^1} = \mathcal{O}(h)$ → better in the sense because function values and gradients are controlled

$\|u-u_h\|_{L^2} = \mathcal{O}(h^2)$ → better in terms of order, because it is 2

Q 7

(8) Maximum principle:

For harmonic functions (e.g. Laplace problem) the maximum is attained on the boundary.

(9) M matrix?

Alles | Leerzeichen einfügen

A: The inverse A^{-1} of a matrix is component-wise positive.

(10) With (9) we obtain a discrete maximum principle

Important because: property from continuous level is conserved after discretization

11 Examples of elliptic, parabolic, hyperbolic:

$$-\Delta u = f$$

$$\partial_t u - \Delta u = f$$

$$\partial_{tt} u - \Delta u = f$$

Characterization: matrix of coefficients of highest derivative:

elliptic: all eigenvalues have the same sign

parabolic: " , but only one is zero

hyperbolic: all same sign, but only one negative sign.

Lit: Chapter 4

17 Well-posed results

A: Lax-Milgram for elliptic problems

Q: Assumptions?

Abstract problem: Find $u \in V$: $a(u, \varphi) = \ell(\varphi) \; \forall \varphi \in V$

i) $|\ell(\varphi)| \leq C \|\varphi\|$

ii) $|a(u,\varphi)| \leq \gamma \|u\| \cdot \|\varphi\|$

iii) $a(u,u) \geq \alpha \|u\|^2$

→ existence, uniqueness, stability

21 Master element

One advantage is that we formulate numerical quadrature on only one element and compute all element integrals by only transforming them to this master element.

(23) + (24) s.o.

(27) + (28) Best approximation + Galerkin orthogonality

↓

Céa's lemma

$$\|u-u_h\| \le c\,\|u-\varphi_h\| \quad \text{for all } \varphi_h \in V_h$$

Proof.

$$\alpha\|u-u_h\|^{\cancel{2}} \le a(u-u_h,u-u_h)$$

$$= a(u-u_h,\, u-\varphi_h+\varphi_h-u_h)$$

$$= a(u-u_h,\, u-\varphi_h) + \underbrace{a(u-u_h,\, \varphi_h-u_h)}_{=0 \text{ thanks to Galerkin orthogonality}}$$

$$\le \gamma\,\cancel{\|u-u_h\|}\,\|u-\varphi_h\|$$

$$\Rightarrow \|u-u_h\| \le \frac{\gamma}{\alpha}\|u-\varphi_h\|$$

$\blacksquare$ end proof

Q: Céa's lemma is only a qualitative result

How can ~~we obtain quantitative~~ estimates in the mesh

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Q: Céa's lemma is only a qualitative result

How can we obtain quantitative estimates in the mesh size h?

A: Construct interpolation error estimates and use them with Céa's lemma to obtain convergence error estimates.

↳ see Q17 above.

XXX What is the idea of multigrid?

↳ Solve problem on different meshes with simple iterative methods. Combine solutions afterward.

↳ Why working?

A: High frequency eigenvalues are damped on fine meshes, while smaller eigenvalues are damped on coarse meshes

45 Discretization of time-dependent problems?

↳ Rothe's method: First time (FD) and then space with FEM.

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VIII End

! Thank you very much !

All the best and see you again at some point

Von meinem iPad gesendet
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