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Numerics of Partial Differential Equations – Tutorial 12

Exercise 12.1

Given is the wave equation in its strong form

$$
\begin{aligned}
\rho \partial_{tt}^2 u - \nabla \cdot (\nabla u) &= f \text{ in } \Omega \times I \\
u &= 0 \text{ on } \partial\Omega \times I \\
u &= u_0 \text{ on } \Omega \times \{t = 0\} \\
\partial_t u &= v_0 \text{ on } \Omega \times \{t = 0\}
\end{aligned}
$$

(a) Discretize the problem in time using the one-step-theta method.

(b) Derive the weak form for the time-discretized problem. Choose an appropriate function space.

(c) We choose $\theta = 1$. Can we show existence in time step t^n with a given solution on the previous time steps?

Exercise 12.2

Let us consider the weak formulation of the wave equation after time-discretization via the implicit trapezoidal rule (step size k).

$$
\begin{aligned}
(v^n - v^{n-1}, \varphi) + \frac{1}{2} k (\nabla u + \nabla u^{n-1}, \nabla \varphi) &= \frac{1}{2} (f^n + f^{n-1}, \varphi) && \forall \varphi \in H_0^1 \\
(u^n - u^{n-1}, \psi) - \frac{1}{2} k (v^n + v^{n-1}, \psi) &= 0 && \forall \psi \in L^2
\end{aligned}
$$

Show that in a closed system ($f^n = 0$, $n = 1, \ldots, N$)

$$
\underbrace{\|\nabla u^n\|^2}_{E_{\text{pot}}^n} + \underbrace{\|v^n\|^2}_{E_{\text{kin}}^n} = \underbrace{\|\nabla u^{n-1}\|^2}_{E_{\text{pot}}^{n-1}} + \underbrace{\|v^{n-1}\|^2}_{E_{\text{kin}}^{n-1}}
$$

holds, therefore, we have conservation of energy for the system.

Home Assignment 12.3

Let A be an elliptic operator. We consider the fractional-step-*theta* scheme with parameters θ, α, β and $\theta' = 1 - 2\theta$.

$$
u^{n+1} = (I + \alpha\theta k A)^{-1} (I - \beta\theta k A)(I + \beta\theta' k A)^{-1} (I - \alpha\theta' k A)(I + \alpha\theta k A)^{-1} (I - \beta\theta k A) u^n
$$

(a) Which conditions must hold, so that the scheme has a consistency order of 1 or 2.

(b) For which choice of α, β, θ is the scheme of consistency order 2 and strongly A-stable?