Institut für Angewandte Mathematik
Leibniz Universität Hannover

Prof. Dr. Thomas Wick
M.Sc. Philipp Thiele

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Numerics of Partial Differential Equations – Tutorial 2

Exercise 2.1 [Classification I]

Classify the following differential equations with respect to order, linearity, coupling and whether they are scalar valued or a differential equation system. Let $\Omega \subset \mathbb{R}^n$. Please explain your answers.

(a) Find $u : \Omega \to \mathbb{R}$:

$$-\Delta u + \partial_x u = f$$

(b) Find $u : \Omega \to \mathbb{R}$:

$$\partial_{xx} u + \partial_{tx} u = g$$

(c) Find $u : \Omega \to \mathbb{R}$:

$$-\Delta u + u^2 = f$$

(d) Find $v : \Omega \to \mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$:

$$\partial_t v + (v \cdot \nabla) v - \frac{1}{Re} \Delta v + \nabla p = f, \quad \nabla \cdot v = 0$$

(e) Find $u : \Omega \to \mathbb{R}$ and $\varphi : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\Delta u &= f(\varphi), \\ |\nabla u|^2 - \Delta \varphi &= g(u) \end{aligned}$$

Home Assignment 2.2 [Classification II]

Classify the following differential equations with respect to order, linearity, coupling and whether they are scalar valued or a differential equation system. Let $\Omega \subset \mathbb{R}^n$. Please explain your answers.

(a) Find $u : \Omega \to \mathbb{R}$:

$$\partial_{xy} u + \partial_y u + u^2 = 0$$

(b) Find $u : \Omega \to \mathbb{R}$:

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f$$

(c) Find $v : \Omega \to \mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$:

$$\partial_t v + (v \cdot \nabla) v + \nabla p = f, \quad \nabla \cdot v = 0$$

(d) Find $u : \Omega \to \mathbb{R}$ and $\varphi : \Omega \to \mathbb{R}$:

$$\begin{aligned} -\nabla \cdot (a(\varphi) \nabla u) &= f, \\ a(\varphi) |\nabla u|^2 - \Delta \varphi &= g \end{aligned}$$

(e) Let Ω_1 and Ω_2 be given, with $\Omega_1 \cap \Omega_2 = 0$ and $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma$ and $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \Omega$.
Find $u_1 : \Omega_1 \to \mathbb{R}$ and $u_2 : \Omega_2 \to \mathbb{R}$:

$$\begin{aligned} -\Delta u_1 &= f_1 \quad \text{in } \Omega_1, \\ -\Delta u_2 &= f_2 \quad \text{in } \Omega_2, \\ u_1 &= u_2 \quad \text{on } \Gamma, \\ \partial_n u_1 &= \partial_n u_2 \quad \text{on } \Gamma, \end{aligned}$$

Home Assignment 2.3 [Well-posedness of the Poisson problem]

(a) Integrate the Poisson problem explicitly and derive the formulation

$$u(x) = -\int_0^x \left(\int_0^t f(s)ds \right) dt + C_1 x + C_2.$$

(b) Derive C_1 and C_2 explicitly.

(c) Show that

$$u(x) = \int_0^L G(x,s) f(s) ds + a\frac{(L-x)}{L} + b\frac{x}{L}$$

solves the Poisson problem.