Institut für Angewandte Mathematik
Leibniz Universität Hannover

Prof. Dr. Thomas Wick
M.Sc. Philipp Thiele

Hannover, 02.-06.11.2020

Numerics of Partial Differential Equations – Tutorial 3

Exercise 3.1

Let $T > 0$ and $\nu > 0$. Consider the domain $\Omega := \mathbb{R}$ and the time interval $I := (0, T)$. Furthermore, let $u^0 \in C^0(\Omega)$ be a 1-periodic function, i.e. $u^0(0) = u^0(1)$.
Find $u(x,t)\colon \Omega \times I \to \mathbb{R}$ such that

$$\partial_t u(x,t) - \nu \partial_{xx} u(x,t) = 0 \text{ in } \Omega \times I$$
$$u(x,0) = u^0(x) \text{ in } \Omega \times \{0\}$$

We assume that the problem is well-posed.
Discretization:
Let $\theta \in [0,1]$. Let $N \in \mathbb{N}$ be the number of time steps and $J \in \mathbb{N}$ be the number of inner grid points. From this we get the time step size $k = \frac{T}{N}$ and the mesh size $h = \frac{1}{J}$, as well as the discrete time points $t^n = nk$ $(0 \le n \le N)$ and grid points $x_j = jh$ $(-1 \le j \le J+1)$.
We denote our discrete solution as $U_j^n := U(x_j, t^n)$ and get the following discrete problem

$$\frac{U_j^{n+1} - U_j^n}{k} - \theta\nu \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} - (1-\theta)\nu \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = 0$$

for all $0 \le n \le N-1$ and for all $1 \le 1 \le J$.
Initial and boundary conditions:
Due to the 1-periodicity, we have the following boundary conditions

$$\forall 0 \le n \le N-1, \quad U_0^n = U_J^n, \quad U_{J+1}^n = U_1^n.$$

As our initial condition $(n = 0)$ we choose

$$\forall 1 \le j \le J, \quad U_j^0 = u^0(x_j).$$

Notation:
To simplify the notation we introduce the variable

$$C_d = \nu \frac{k}{h^2}$$

and the discrete solution vector

$$U^n = (U_i^n)_{1 \le i \le J} \in \mathbb{R}^J.$$

(a) Write down the discretization from above in matrix-vector form for $0 \le n \le N-1$:

$$(B_I^\theta)^{-1} U^{n+1} = B_E^\theta U^n$$

with

$$B_I^\theta = (I + \theta C_d B)^{-1}, B_E = I - (1-\theta) C_d B$$

where $I \in \mathbb{R}^{J \times J}$ is the identity matrix and $B \in \mathbb{R}^{J \times J}$ is a quadratic matrix. Furthermore explain the entries of the matrix B.

(b) What do the matrices B_I^θ and B_E^θ look like in the edge cases $\theta = 0$ and $\theta = 1$?

(c) Explain the existence of B_I^θ.

Home Assignment 3.2

Let the non-stationary heat equation be given:

$$
\begin{aligned}
(\partial_t - \partial_{xx})u &= f \text{ in } Q = (0,T) \times \Omega \\
u &= 0 \text{ auf } \Sigma = (0,T) \times \partial\Omega \\
u(0,x) &= u_0(x) \text{ in } \Omega
\end{aligned}
$$

(a) Use the One-step-θ scheme (linear interpolation between forward and backward Euler, also known as Crank-Nicholson) to discretize the PDE in time.

(b) Which type of problem do we get after this kind of discretization?

(c) Use the central difference quotient for the space discretization of this problem.

Home Assignment 3.3

Let the non-stationary heat equation be given:

$$
\begin{aligned}
(\partial_t - \partial_{xx})u &= f \text{ in } Q = (0,T) \times \Omega \\
u &= 0 \text{ auf } \Sigma = (0,T) \times \partial\Omega \\
u(0,x) &= u_0(x) \text{ in } \Omega
\end{aligned}
$$

(a) Use the central difference quotient for the space discretization of this problem.

(b) Which type of problem do we get after this kind of discretization?

(c) Use the forward Euler scheme to discretize the PDE in time.

(d) Use the backward Euler scheme to discretize the PDE in time.