Institut für Angewandte Mathematik
Leibniz Universität Hannover
Prof. Dr. Thomas Wick
M.Sc. Philipp Thiele

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Numerics of Partial Differential Equations – Tutorial 4

Exercise 4.1 [Formulations I]

(a) Let $\Omega \subset \mathbb{R}^n$, $a\colon \Omega \to \mathbb{R}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.
Find $u\colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla(a(x)\nabla u(x)) + b\nabla u(x) + cu(x) &= f \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on} \quad \Gamma = \partial\Omega \end{aligned}$$

Derive the weak form of this problem.

(b) Let $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ (sufficiently smooth) and V:

$$V := H_0^1(\Omega) = \{v \in L^2(\Omega) |\ \partial_{x_i} v \in L^2(\Omega)\ \forall i = 1, \dots, n\ ,\ v = 0 \text{ on } \partial\Omega\}.$$

Furthermore let $b : \Omega \to \mathbb{R}^n$ be a vector field. Find $u \in V$ such that

$$(\nabla u, \nabla\phi) + (b\nabla u, \phi) = (f, \phi) \quad \forall \phi \in V.$$

(i) Derive the strong form (D).

(ii) Derive (if possible; explanation) the energy minimization formulation (M).
Discuss the result extensively.

Exercise 4.2 [Shape functions in 1D]

In this exercise the linear and quadratic shape functions for the general one dimensional element $[x_0, x_1]$ should be derived.
Use, where it is possible, $h = x_1 - x_0$ to simplify your calculations. Derive the monomial representation of your results.

(a) Derive the linear shape functions $\varphi_0, \varphi_1 \in P_1$ through Lagrange interpolation.
Hint: The linear shape functions have the following properties:

$$\varphi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(b) For the quadratic shape functions $\psi_i \in P_2, i = 0, 1, 2$ an additional node $x_2 = (x_0 + x_1)/2$ is required. Derive the shape functions through Newton interpolation.
Hint: The quadratic shape functions have similar properties as in (a).

Home Assignment 4.3 [Formulations II]

(a) Let $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ (sufficiently smooth). Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Derive (if possible; explanation) the energy minimization formulation (M).

(b) Let $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ (sufficiently smooth). Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

(i) Derive the weak form (V).

(ii) Derive (if possible; explanation) the energy minimization formulation (M). Discuss the result extensively.

Home Assignment 4.4 [Formulations III]

We consider the non-stationary heat equation after time discretization with the One-step-θ scheme (result from 3.2(a)).
Derive the weak form of this PDE.