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Numerics of Partial Differential Equations – Tutorial 9

Exercise 9.1

We have the following problem with Neumann boundary conditions:
Let $\Omega \subset \mathbb{R}^n$ be bounded, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Then the problem reads:
Find u, such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u &= g \quad \text{auf } \partial\Omega. \end{aligned}$$

For the existence of a solution we need the compatibility condition

$$\int\limits_{\Omega} f(x)\, dx + \int\limits_{\partial\Omega} g(x)\, ds = 0.$$

Now check whether the variational formulation fulfills the assumptions of the Lax-Milgram theorem and therefore has (except a constant) an exact solution.

Home Assignment 9.2

We have the following problem with Dirichlet boundray conditions:
Let $\Omega \subset \mathbb{R}^n$ be bounded and $f \in L^2(\Omega)$. Then the problem reads:
Find u, such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= h \quad \text{auf } \partial\Omega. \end{aligned}$$

We assume that a function $u_0 \in H^1$ exists with

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= h \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore the problem in variational form reads: Find $u \in H_0^1$, such that

$$(\nabla \tilde{u}, \nabla \phi) = (f, \phi) - (\nabla u_0, \nabla \phi)\ \forall \phi \in H_0^1$$

We can write the solution u to this inhomogeneous Dirichlet problem as

$$u = u_0 + \tilde{u}.$$

Now check whether the variational formulation fulfills the assumptions of the Lax-Milgram theorem and therefore a unique solution exists.

Home Assignment 9.3

We have the following problem with Robin boundary conditions. Let $\Omega \subset \mathbb{R}^n$ be bounded, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $c \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$. The problem reads:
Find u, such that

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega, \\ \partial_n u + bu &= h \quad \text{auf } \partial\Omega. \end{aligned}$$

In variational form the problem reads:
Find $u \in H^1$, such that

$$
\begin{aligned}
a(u,\phi) &= l(\phi) \quad \forall \phi \in H^1 \\
\text{with } a(u,\phi) &= \int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega c u \phi \, dx + \int_{\partial\Omega} b u \phi \, ds, \\
l(\phi) &= \int_\Omega f v \, dx + \int_{\partial\Omega} g \phi \, ds.
\end{aligned}
$$

For this we also need an additional condition:

Let Ω be a bounded C^1-Domain and let $c \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$ be almost everywhere non negative with the condition

$$
\int_\Omega c^2 \, dx + \int_{\partial\Omega} b^2 \, ds > 0.
$$

Check whether the variational formulation fulfills the assumptions of the Lax-Milgram theorem and therefore an unique solution to the problem exists.