Master M2 Optimisation

par Grégoire Allaire, Antonin Chambolle, Thomas Wick

Basic course of the Master Program "Mathematical Modelling"
Ecole Polytechnique, ENPC and Pierre et Marie Curie University
Academic year 2016-2017

Examen 5, Oct 13, 2016
in Amphi Lagarrique

Exercise 1

1. Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$. Compute the Gâteaux derivative of

$$f(u) = \|u\|_H^3.$$

Answer of exercise 1

First we recognize that

$$f(u) = \|u\|_H^3 = (\sqrt{(u,u)_H})^3,$$

where $(u,u)_H := \int_\Omega |u|^2$. Then we simply use the formulae as in the exercise or one can compute in a very similar way like in school calculus (using directly the chain rule and differentiating). The direction into which we differentiate is $h \in H$, such that we obtain

$$f'(u)h = ((\sqrt{(u,u)_H})^3)'(h) = 3\sqrt{(u,u)}^2 \cdot \frac{1}{2\sqrt{(u,u)}} \cdot 2(u,h) = 3\|u\|_H \cdot (u,h).$$

Exercise 2

Application of the SQP method. Let $f(x,y) = -x - 0.5y^2$ and $c(x,y) = 1 - x^2 - y^2$ be given.

1. Formulate the Newton-KKT iteration.
   Hint: Construct the Lagrangian function, formulate the KKT conditions, and formulate then the Newton-KKT system.

Answer of exercise 2

We are given:

$$f(x,y) = -x - 0.5y^2, \quad c(x,y) = 1 - x^2 - y^2.$$

- Lagrangian:

$$L(x,y,\lambda) = f(x) - \lambda c(x) = -x - 0.5y^2 - \lambda(1 - x^2 - y^2)$$

- First order KKT conditions:

$$
\begin{aligned}
L'_x(x,y,\lambda) &= -1 + 2\lambda x \\
L'_y(x,y,\lambda) &= -y + 2\lambda y \\
L'_\lambda(x,y,\lambda) &= -(1 - x^2 - y^2)
\end{aligned}
$$

  which yields the first order optimality condition:

$$
F(x,y,\lambda) = \begin{pmatrix} -1 + 2\lambda x \\ -y + 2\lambda y \\ -(1 - x^2 - y^2) \end{pmatrix}
$$

  . Thus we have to solve $F(x,y,\lambda) = 0$. And this solution is obtained with a Newton scheme that is discussed next.

- Newton-KKT system

  1. Newton-KKT matrix (yet without Newton iteration indices):

$$
F'(x,y,\lambda) = \begin{pmatrix} 2\lambda & 0 & 2x \\ 0 & -1 + 2\lambda & 2y \\ 2x & 2y & 0 \end{pmatrix}
$$

     .

  2. Newton iteration: Given a start guess (x_0, y_0, λ_0) we iterate for $k = 1, 2, 3, \ldots$ such that the following defect-correction scheme is solved:

$$
\begin{aligned}
F'(x_k, y_k, \lambda_k)(\delta x_k, \delta y_k, \delta \lambda_k)^T &= -F(x_k, y_k, \lambda_k), \\
(x_{k+1}, y_{k+1}, \lambda_{k+1})^T &= (x_k, y_k, \lambda_k)^T + (\delta x_k, \delta y_k, \delta \lambda_k)^T
\end{aligned}
$$

     Specifically the linear equation system reads:

$$
\begin{pmatrix} 2\lambda & 0 & 2x_k \\ 0 & -1 + 2\lambda & 2y_k \\ 2x_k & 2y_k & 0 \end{pmatrix} (\delta x_k, \delta y_k, \delta \lambda_k)^T = - \begin{pmatrix} -1 + 2\lambda x_k \\ -y_k + 2\lambda y_k \\ -(1 - x_k^2 - y_k^2) \end{pmatrix}
$$

     .

S'il vous plait, tourner la page pour la version francaise.

Exercise 1

1. Soit H un espace Hilbert avec le produit scalaire $(\cdot,\cdot)_H$ et la norme $\|\cdot\|_H$. Calculer la dérivée au sens de Gâteaux du

$$f(u) = \|u\|_H^3.$$

Answer of exercise 1

Voir la version anglaise.

Exercise 2

On veut appliquer la méthode SQP (sequentiel quadratic programming). Soit $f(x,y) = -x - 0.5y^2$ et $c(x,y) = 1 - x^2 - y^2$

1. Formuler l'iteration au sens de Newton-KKT (KKT = Karush-Kuhn-Tucker).
   Conseil: Construire la fonction Lagrangian, formuler les conditions de KKT, et formuler le systéme final.

Answer of exercise 2

Voir la version anglaise.