Master M2 Optimization

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http://www.cmap.polytechnique.fr/~wick/m2_fall_2016_engl.html

Exercise 1

Let $[a, b]$ a closed interval on R.

1. Show that the function space $C[a, b]$ is complete.

Answer of exercise 1

Let the metric defined by

$$d(x, y) := \max_{t \in [a,b]} |x(t) - y(t)|.$$

Let $\{x_k\}_k$ be any Cauchy sequence in $C[a, b]$. For a given $\varepsilon > 0$, there is an N such that for all $n, m > N$ we have

$$d(x_m, x_n) := \max_{t \in [a,b]} |x_m(t) - x_n(t)| < \varepsilon.$$

For any fixed $t := t_0 \in [a, b]$ we obtain

$$|x_m(t_0) - x_n(t_0)| < \varepsilon.$$

Thus we have constructed a Cauchy sequence $\{x_k(t_0)\}_k$ of real numbers. But R is complete and consequently this sequence converges, for instance

$$x_m(t_0) \to x(t_0) \quad \text{as } m \to \infty.$$

We repeat the entire procedure for other $t_1, t_2, \ldots$ and can thus associate to each $t \in [a, b]$ a unique real number $x(t)$. Therefore, we have constructed a pointwise limit function x on $[a, b]$. It remains to show that, firstly, $x \in C[a, b]$ and secondly $x_m \to x$. From $d(x_m, x_n)$ we obtain for $n \to \infty$:

$$\max_{t \in [a,b]} |x_m(t) - x(t)| \le \varepsilon.$$

This holds for every $t \in [a, b]$ and therefore

$$|x_m(t) - x(t)| \le \varepsilon.$$

Consequently, we have shown the 2nd part that x_m uniformly converges to x. In addition all x_m are continuous on $[a, b]$. Thus from calculus we know that the limit function x of a uniformly converging sequence $\{x_k\}_k$ is itself continuous, i.e., $x \in C[a, b]$. Consequently, everything has been shown.
Q.E.D.

Exercise 2

1. Show that strong convergence implies weak convergence.
2. Give a counter-example that the opposite is in general not true. [**Hint:** One possibility is to work with a orthonormal sequence $\{e_k\}_k$ in a Hilbert space.]

Answer of exercise 2

Answer to 1): By definition $x_n \to x$ means $\|x_n - x\| \to 0$. Thus for every linear functional $f \in X'$ we compute:

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\| \, \|x_n - x\| \tag{1}$$

Because of strong convergence the last term tends to zero and thus, $x_n \rightharpoonup x$.
Answer to 2): One possibility is to work with an orthonormal sequence $\{e_n\}_n$ in a Hilbert space H. Here, every $f \in H'$ has a Riesz representation $f(x) = \langle x, z\rangle$ for $x \in H$. In particular, it holds $f(e_n) = \langle e_n, z\rangle$. In a Hilbert space the Bessel inequality holds true:

$$\sum_{n=1}^{\infty} |\langle e_n, z\rangle|^2 \leq \|z\|^2.$$

This means that the infinite series on the left converges and therefore, the terms $|\langle e_n, z\rangle|$ must tend to zero for $n \to \infty$. Then it can be inferred that

$$f(e_n) = \langle e_n, z\rangle \to 0.$$

Since f was arbitrary, the assertion is shown and $e_n \rightharpoonup 0$. On the other hand, $\{e_n\}_n$ does not converge strongly because

$$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n\rangle = 1 + 1 = 2.$$

Thus no convergence in the strong norm.

Exercise 3

Let $(X, \|\cdot\|)$ be a Banach space.

1. Then $f(x) := \|x\|$ is weakly lower semi-continuous.

Answer of exercise 3

To answer this exercise, different ways are possible. To recall, we must show the weak lower semi-continuity:

$$x_n \rightharpoonup x \quad \Rightarrow \quad f(x) \leq \liminf_{n\to\infty} f(x_n) \quad \text{for } n \to \infty.$$

Possibility 1 (is trivial if we assume convexity and a theorem from functional analysis): Each continuous functional f on a Banach space X that is in addition convex, is weakly lower semi-continuous. The norm is clearly continuous (follows because the metric $d(x, y)$ is continuous on a normed space). The convexity follows from

$$\|\lambda x_1 + (1-\lambda)x_2\| \leq \lambda\|x_1\| + (1-\lambda)\|x_2\| \quad \text{for all} \lambda \in [0,1].$$

Thus the functional $f(x) := \|x\|$ is continuous and convex and therefore the theorem applies and therefore the assertion is shown.
Remark: This answer shows how important convexity is for such problems and in general for optimization.

Possibility 2 (works without convexity but also needs consequences of the Hahn-Banach theorem from functional analysis, which is however very standard and well known: Let $(X, \|\cdot\|)$ be a linear normed space with $x_0 \in X$. Then there exists a linear continuous functional $f \in X'$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

We work with a contradiction proof. Assume that $\|x_0\| > \liminf_{n\to\infty} \|x_n\|$. Then there exists a number c such that

$$\|x_0\| > c > \lim \inf_{n\to\infty} \|x_n\|.$$

Thus it exists a subsequence $\{x_{n_k}\}_k$ such that

$$\|x_0\| > c > \|x_{n_k}\|.$$

We now show that the right inequality satisfies weak convergence:

$$|f(x_{n_k})| \leq \|f\|\,\|x_{n_k}\| = \|x_{n_k}\| < c,$$

because according to Hahn-Banach a functional with $\|f\| = 1$ exists. Thus we have weak convergence on the one hand:

$$f(x_0) = \lim_{n_k\to\infty} f(x_{n_k}) \leq c. \tag{2}$$

In the second part we show now that the left inequality $\|x_0\| > c$ holds true. Hahn-Banach yields as well $f(x_0) = \|x_0\| > c$. But this result contradicts (2) and therefore $f(x) := \|x\|$ must be weakly lower semi-continuous.
Q.E.D.

S'il vous plait, tourner la page pour la version francaise.

Exercise 1

Soit $[a, b]$ un intervalle fermé sur R.

1. Montrer que l'espace $C[a, b]$ est complet.

Answer of exercise 1

Exercise 2

1. Montrer que convergence forte implique convergence faible.
2. Donner un contre-exemple que le contraire n'est pas vrai. [**Conseil:** Une possibilité est de travailler avec une séquence orthonormé $\{e_k\}_k$ dans un espace de Hilbert.]

Answer of exercise 2

Exercise 3

Soit $(X, \|\cdot\|)$ un espace de Banach.

1. Montrer que $f(x) := \|x\|$ est faiblement semi-continue inférieurement.

Answer of exercise 3