Master M2 Optimization

by Grégoire Allaire, Antonin Chambolle, Thomas Wick

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in Amphi Grégory
Online version on:

http://www.cmap.polytechnique.fr/~wick/m2_fall_2016_engl.html

Exercise 1

Show that a convex subset $A \subset X$ of a Banach space X is weakly sequentially closed if and only if A is closed.

Answer of exercise 1

We first recall that 'sequentially closed' is equivalent to 'closed' in a metric space. Therefore, in our situation (as always in this optimization class) there is no difference between the two notations.

We first state that weakly-closedness implies closedness because the weak topology is 'weaker' than the strong one, and thus a weakly closed subset is a fortiori closed.

For the other direction we employ again the Hahn-Banach theorem and its geometrical interpretation, that a convex and closed subset can be seperated from another set.

We work with a contradiction: Assume that $\{u_n\}_n \subset A$ and the limit $u \in X \setminus A$. From our assumptions we have $u_n \rightharpoonup u$ for $n \to \infty$. Since A is convex and closed it follows with Hahn-Banach that a functional $f \in X'$ and $\lambda \in \mathbb{R}$ exist such that $Ref(u_n) \leq \lambda$ and $Ref(u) > \lambda$. But this means that

$$\lim_{n \to \infty} f(u_n) \neq f(u)$$

which contradictes the weak convergence. In consequence, A must be weakly closed.
Q.E.D.

Exercise 2

Show that a convex functional $F : X \to R$ of a Banach space X is weakly lower-semi continuous if and only if F is lower-semi continuous. [**Hint:** Apply the previous exercise to the epigraph.]

Answer of exercise 2

We know that $f \in X'$ is lsc if and only if $epi(f)$ is closed. We also know that $f \in X'$ is convex and lsc if and only if $epi(f)$ is convex plus closed. From

Exercise 1 we use that $epi(f) = A$ and $epi(f)$ is convex and closed. Thus it follows that $epi(f)$ is weakly closed if and only if f is convex and wlsc.
Related literature online can be for example found on

`https://www.math.uh.edu/~rohop/Fall_11/index.html`

in chapter 5.

Exercise 3

Let us work again in R^n (finite dimensions) for simplicity. Show that when f is convex, any local minimizer x^* is a global minimizer of f. If additionally f is differentiable, then any stationary point x^* is a global minimizer of f.

Answer of exercise 3

This exercise has been taken from Nocedal and Wright: *Numerical optimization, Springer, 2006* on page 16.
Suppose that x^* is a local but not a global minimizer. Then $f(z) < f(x^*)$ for $z \in R^n$. Now we can construct a line segment

$$x = \lambda z + (1-\lambda)x^*, \quad \lambda \in (0,1].$$

For the convexity property of f we have:

$$f(x) = f(\lambda z + (1-\lambda)x^*) \leq \lambda f(z) + (1-\lambda) f(x^*) < f(x^*).$$

The last strict inequality holds because we assume $f(z) < f(x^*)$. Now in any neighborhood of x^* we can construct such a line segment in which the previous inequality is satisfied. But this is now a contradiction to the convexity property and this x^* is not only a local but a global minimizer.

The answer to the 2nd part is as follows. Suppose again that x^* is not a global minimizer. From convexity and differentiability we obtain the following calculation:

$$\begin{aligned}
\nabla f(x^*)^T(z-x^*) &= \frac{d}{d\lambda} f(x^* + \lambda(z-x^*))|_{\lambda=0} && (1)\\
&= \lim_{\lambda\to 0} \frac{f(x^* + \lambda(z-x^*)) - f(x)}{\lambda} && (2)\\
&\leq lim_{\lambda\to 0} \frac{\lambda f(x^*) + (1-\lambda) f(z-x^*) - f(x)}{\lambda} && (3)\\
&= f(z) - f(x^*) && (4)\\
&< 0 && (5)
\end{aligned}$$

Thus:

$$\nabla f(x^*)^T \neq 0$$

for $z \neq x^*$ and consequently x^* is not a stationary point, which is a contradiction to our assumption. And thus x^* is really a global minimizer.
Q.E.D.

Exercise 4

Let $f : X \to R$ be of C^2 type on the open, convex, set $X \subset R^n$.

1. Show that f is convex if and only if the Hessian $\nabla^2 f(x)$ is positive semi-definite for all $x \in X$, which means

$$p^T \nabla^2 f(x) p \geq 0 \quad \forall p \in R^n, \quad \forall x \in X.$$

Answer of exercise 4

$\Rightarrow$ Let f be convex and $x \in X$ and $p \in R^n$. Since X is an open set is follows that $\tau = \tau(x,p) > 0$ exists with $x + tp \in X$ for all $t \in [0, \tau]$. For $0 \leq t \leq \tau$ we obtain

$$0 \leq f(x+tp) - f(x) - t\nabla f(x)^T p = \frac{t^2}{2} p^T \nabla^2 f(x) p + o(t^2). \qquad (6)$$

Multiplication with $\frac{2}{t^2}$ and taking the limit $t \to 0$ yields the assertation.

$\Leftarrow$ Let $x, y \in X$. Taylor expansion for $\sigma \in [0,1]$ yields

$$f(y) - f(x) = \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(x + \sigma(y-x))(y-x) \geq \nabla f(x)^T (y-x) \qquad (7)$$

This is clearly convexity because

$$\frac{f(y) - f(x)}{y - x} \geq \nabla f(x)^T.$$

Q.E.D.