Master M2 Optimization

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Basic course of the Master Program "Mathematical Modelling"
Ecole Polytechnique, ENPC and Pierre et Marie Curie University
Academic year 2016-2017

Exercise meeting 4, Sep 29, 2016

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Online version on:

`http://www.cmap.polytechnique.fr/~wick/m2_fall_2016_engl.html`

Exercise 1

Let $\Omega \subset R^3$ and $u \in H^1(\Omega)$. Consider the functional

$$F(u) = \int_\Omega |\nabla u|^2 + W(u(x))dx$$

with for instance $W(t) = (1 - t^2)^2$.

1. Is $F(u)$ convex?
2. Is $F(u)$ lower semicontinuous? [**Hint:** You can assume that for $u_n \rightharpoonup u$ in $H^1(\Omega)$, it holds

$$\int_\Omega |\nabla u|^2 dx \le \liminf_n \int_\Omega |\nabla u_n|^2 dx$$

which is deduced for instance from the fact that

$$\int_\Omega |\nabla u|^2 dx = \sup_{v \in L^2(\Omega;\mathbb{R}^N)} \int_\Omega v \cdot \nabla u dx - \frac{1}{4}\int_\Omega |v|^2 dx$$

hence it is the sup of weakly continuous linear functionals.]

Answer of exercise 1

We first study the convexity. The functional $F(u)$ cannot be convex because the part $W(u)$ is non-convex. This is clearly seen by two times differentiating $W(u)$ and checking whether $W(u) \ge 0$. Strictly speaking one must use directional derivatives (because the argument of W is a function itself and not a variable!) and first differentiate into direction δu_1 and then into δu_2:

$$W(u) = (1 - u^2)^2,$$
$$W'(u)(\delta u_1) = -(4u - 4u^3)\delta u_1,$$
$$W''(u)(\delta u_2, \delta u_1) = -(4\delta u - 12u^2\delta u_2)\delta u_1.$$

Here we see that $W''(u)$ is not always positive, thus W cannot be convex. But for simplicity we can just take the derivative of $W(t)$:

$$W(t) = -4(1 - 3t^2).$$

We see that for all t, $W(t)$ is not always positive. Thus the function cannot be convex.

With regard to question 2, we need to check if $F(u)$ is weakly lower semi-continuous in H^1. That is

$$u_n \rightharpoonup u \quad \text{in } H^1 \quad \Rightarrow F(u) \leq \liminf_n F(u_n).$$

Now we discuss lower semicontinuity seperately for

$$H_1(u) := \int_\Omega |\nabla u|^2 \, dx,$$
$$H_2(u) := \int_\Omega W(u) \, dx.$$

To show the lower semi-continuity for the combined functional, we use relation what we already had two weeks ago: If H_1 and H_2 are w.lsc. then it follows that

$$F(u) = (H_1 + H_2)(u) \leq \liminf_n H_1 + \liminf_n H_2 \leq \liminf (H_1 + H_2)(u).$$

For H_1 we already had the hint on the exercise sheet.

Let us now look in more detail to $H_2(u)$. We have a lower bound for $F(u)$ by

$$F(u) = \int_\Omega [|\nabla u|^2 + W(u)] \, dx, \tag{1}$$
$$\geq \int_\Omega [|\nabla u|^2 + u^2 - 3] \, dx, \tag{2}$$
$$= \|u\|_{H^1}^2 - b, \tag{3}$$

where $b = \int_\Omega 3 \, dx$. We notice that also $W(u) \geq u^2 - 2$ could have worked, but not $W(u) = u^2 - 1$!!.

Next, we have to justify that there exists a subsequence $\{u_{n_k}\}_k$. This can be inferred from the following considerations.

- First we see that $H_2(u)$ (and specifically $F(u)$) is bounded from below:

  $$\inf_{u \in H^1} F(u) > -\infty.$$

  This is clearly to see because $|\nabla u|^2$ and $W(u)$ are both bounded from below.

- Consequently, there exists a minimizing sequence $\{u_n\}_n$ such that $F(u_n) < \infty$. This can be justified in more detail with the boundedness of a minimizing sequence in terms of coercivity. A sufficient criterion is that a functional $F(u)$ is coercive if

  $$F(u_n) \to \infty \quad \text{if } \|u_n\| \to \infty.$$

  Thus, the sequence $\{u_n\}$ is always bounded by $F(u_n)$.

- In our case we have:

$$F(u) \geq \|u\|_{H^1}^2 - b, \tag{4}$$

which shows coercivity.

- Since $H^1(\Omega)$ is a Hilbert space (and thus reflexive), the functional $F(u)$ is proper (i.e., $F(u) \neq \infty$) and coercive, then each minimizing sequence $\{u_n\}$ for F has a weakly convergent subsequence. Thus there exists an element $u \in H^1$ such that

$$u_{n_k} \rightharpoonup u \quad \text{in } H^1.$$

Why is this possible? Because of the coercivity of F each minimizing sequence is bounded. Otherwise there would be another minimizing subsequence for which $F(u_{n_k}) \to \infty$. But this contradicts that $\{u_n\}$ is a *minimizing* sequence. Consequently, $\{u_n\}$ is contained in a closed ball, which is weakly compact.

- Therefore, we have constructed a minimizing subsequence $\{u_n\}$ and also a limit $u \in H^1(\Omega)$.

The theorem of Rellich yields now that there exists another subsequence (using the same index) that converges strongly in L^2:

$$u_{n_k} \to u \quad \text{in } L^2.$$

Here we can extract again another subsequence that converges a.e. (almost everywhere). Since W is (obviously) continuous we have:

$$u_{n_k} \to u \quad \text{a.e.} \qquad \Rightarrow \qquad W(u_{n_k}) \to W(u).$$

Thus, Fatou's lemma is applicable and yields:

$$H_2 := \int_\Omega W(u)\,dx \leq \liminf_k \int_\Omega W(u_{n_k})\,dx.$$

Consequently we have shown the w.lsc of H_2 and thus the entire proof is finished.

Exercise 2

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set (or a cube, or a line for instance), let $\theta \in]0,1[$ and consider E_0 a measurable set of volume θ in $[0,1]^N$, and the sets

$$E = \bigcup_{z \in \mathbb{Z}^N} z + E_0, \quad F = \mathbb{R}^N \setminus E.$$

Let $a, b \in \mathbb{R}$ (or in fact any vector space) and $u_n \in L^\infty(\Omega)$ defined by

$$u_n(x) = a\chi_{E/n}(x) + b\chi_{F/n}(x) = a\chi_E(nx) + b\chi_F(nx),$$

where χ is as usual the indicator function. Show that $u_n \rightharpoonup u$ in $L^p(\Omega)$ ($1 \leq p < \infty$) or $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$, where u is the constant function $\theta a + (1-\theta)b$.

Answer of exercise 2

Not yet discussed in the TD.