Master M2 Optimization

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Online version on:

`http://www.cmap.polytechnique.fr/~wick/m2_fall_2016_engl.html`

Exercise 1

Let us study another optimization existence problem in an infinite function space, and which has an immediate practical application: Can we achieve (i.e., does there exist a solution) such that we obtain optimal heating of room of size $\Omega \subset R^3$ with a desired final temperature u_d? Thus we try to minimize the cost functional:

$$\min J(u,q) = \frac{1}{2}\|u-u_d\|^2_{L^2} + \frac{\lambda}{2}\|q\|^2_{L^2}$$

The term of interest is $\|u-u_d\|^2_{L^2}$, thus we want to match the desired temperature. The second term is a so-called (Tikonov) regularization term which is important for both theory as well as numerics, as we later see in the proof. The above cost functional should be minimized with respect to the (stationary) heat equation (i.e., Laplace problem): Find u such that

$$-\Delta u = \beta q \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

where $\beta > 0$ is a constant. Here the important quantity is $q \in L^2(\Omega)$ which is the so-called optimal control. This quantity must be chosen in such a way that $J(u,q)$ is minimized. Several manipulations allow to replace the state variable $u \in H^1(\Omega)$ by the control variable q in terms of a solution operator

$$S : L^2 \to L^2, \quad \text{with } S : q \to u(q)$$

from which we obtain the reduced cost functional:

$$\min f(q) = \frac{1}{2}\|S(q)-u_d\|^2_{H^1} + \frac{\lambda}{2}\|q\|^2_{L^2}.$$

Let us formulate the problem: Let Q and V be Hilbert spaces. Furthermore let Q_{ad} be a bounded, closed, convex subset $Q_{ad} \subset Q$ and let $u_d \in V$ be a given desired state. Let the regularization parameter be $\lambda \geq 0$ and let $S : Q \to V$ be a linear, bounded operator.

1. Show that the quadratic problem

$$\min_{q \in Q_{ad}} f(q) = \frac{1}{2}\|S(q) - u_d\|_V^2 + \frac{\lambda}{2}\|q\|_Q^2,$$

has an optimal solution $\bar{q}$.
2. Show that $\bar{q}$ is unique if $\lambda > 0$ or if S is injective.
3. What are possible solution algorithms to solve $\min f(q)$ with the help of a computer? Please outline the main steps.
4. **Work plan for Question 1:** Please work in the steps as shown in the answer of Exercise 4 from last week; thus, show that an infimum of $f(q)$ exists, justify that a subsequence $\{q_{n_k}\}$ converges weakly to an element $\bar{q} \in Q_{ad}$, show that $f(q)$ is w.lsc., finally show (not in the exercise from last week though), that

$$\liminf_{k \to \infty} f(q_{n_k}) = \inf_{q \in Q_{ad}} f(q)$$

and finally justify that $\inf_{q \in Q_{ad}} f(q) = \inf_{q \in Q_{ad}} f(\bar{q})$.
5. **Work plan for Question 2:** For uniqueness we need to show that $f(q)$ is strictly convex. Please try to work with $f'' > 0$. But be careful with the derivatives in function spaces (see again the answer of Exercise 4 last week).

Answer of exercise 1

Here are the answers to Question 1 and 2 (the third question will be discussed next week).

1. $f(q) \geq 0$ it follows that $\inf_{q \in Q_{ad}} f(q)$ exists.
2. Thus it exists a minimizing sequence $\{q_n\}_n \subset Q_{ad}$ with

$$f(q_n) \to \inf_{q \in Q_{ad}} f(q) \quad (n \to \infty).$$

3. Since $\dim(Q_{ad}) = \infty$ a bounded and closed set is not compact. But Q_{ad} is additionally convex and Q reflexive (because Q is a Hilbert space).
Why does this statement hold true?
We work with Dirk Werner *Funktionalanalysis* (other books in French or English have these two theorems as well):

   - Theorem: In a reflexive space X, each bounded sequence has a weakly convergent subsequence.
   - However the previous statement does not yield whether the limit element is a part of X or not. Here we use a second theorem which yields the weak closedness: Let X be a normed space and $V \subset X$ be closed and convex. For a weakly convergent sequence $\{x_n\}_n \subset V$ with $x_n \rightharpoonup x$ it holds that also $x \in V$, i.e., the space V is weakly closed.

These two previous theorems ensure that Q is weakly compact, which means in particular that a limit $\bar{q} \in Q_{ad}$ exists such that

$$q_{n_k} \rightharpoonup \bar{q} \quad \text{for } k \to \infty.$$

4. Next we need to comment on the continuity of $f(\cdot)$. Of course the norm $\|\cdot\|$ is bounded and also $\|S(q) - u_d\|$ is continuous since the operator S is bounded. But in general we do not have weak continuity, namely:

$$q_{n_k} \rightharpoonup \bar{q} \quad \Rightarrow \quad f(q_{n_k}) \to f(\bar{q})$$

does not hold!! But f is convex and f is continuous which implies w.lsc:

$$q_{n_k} \rightharpoonup \bar{q} \quad \Rightarrow \quad f(\bar{q}) \leq \liminf_{k} f(q_{n_k}).$$

5. In the final step we need to show the equality that the infimum is really taken:

$$\liminf_{k\to\infty} f(q_{n_k}) = \inf_{q \in Q_{ad}} f(q)$$

But this is now trivial because of the closedness of Q_{ad}; thus $\bar{q} \in Q_{ad}$, it can be inferred that

$$\inf_{q \in Q_{ad}} f(q) = f(\bar{q}).$$

Thus $\bar{q}$ is the minimum and there the optimal control that we seeked.
Q.E.D.

Answer to Question 2:
As long as $\lambda > 0$, we always obtain uniqueness. In the case of $\lambda = 0$ we have to hope that S is injective. In order to show uniqueness we show that $f'' > 0$. But the arguments of f are functions thus we work with Gˆteaux derivatives. Let $q, \delta q_1, \delta q_2 \in Q$. We calculate:

$$f'(q)\delta q_1 = (S(q) - u_d, S(\delta q_1))_V + \lambda(q, \delta q_1)_Q,$$
$$f''(q)(\delta q_2, \delta q_1) = (S(\delta q_2), S(\delta q_1))_V + \lambda(\delta q_2, \delta q_1)_Q$$

Choose now $\delta q \in Q_{ad}$ with $\delta q \neq 0$. For $\lambda > 0$ we obtain:

$$f''(q)(\delta q_2, \delta q_1) \geq \lambda(\delta q, \delta q)_Q = \|\delta q\|^2 > 0.$$

Thus f is strictly positive and therefore $\bar{q}$ unique. In the case of $\lambda = 0$ and S injective we obtain:

$$f''(q)(\delta q_2, \delta q_1) = (S(\delta q), S(\delta q))_V = \|S(\delta q)\|_V^2 > 0$$

The last argument holds because of the injectivity of S and that $q \neq 0$ we have $S(\delta q) \neq 0$ and thus $\|S(\delta q)\|_V^2 > 0$. Q.E.D.