Master M2 Optimization

by Grégoire Allaire, Antonin Chambolle, Thomas Wick

Basic course of the Master Program "Mathematical Modelling"
Ecole Polytechnique, ENPC and Pierre et Marie Curie University
Academic year 2016-2017

Exercise meeting 6, Oct 13, 2016

in Amphi Lagarrigue

Online version on:

`http://www.cmap.polytechnique.fr/~wick/m2_fall_2016_engl.html`

Exercise 1

One of the most successful methods for nonlinear constrained optimization is the SQP algorithm (sequentiell quadratic programming). Let us study SQP in R^n in a bit more detail. We are given:

$$\min f(x) \quad \text{s.t. } c(x) = 0,$$

where $f : R^n \to R$ and $c : R^n \to R^m$.

1. Formulate the Lagrangian function $\mathcal{L}(x, \lambda)$
2. State the first order (KKT) conditions $F(x, \lambda)$
3. Based on the KKT conditions, formulate a Newton scheme, a so-called Newton-KKT system
4. Under which conditions is the KKT-matrix in Newton's method well-defined?

Exercise 2

1. Let $V = C[0, 1]$ and define $f : V \to R$ via

   $$f(u(x)) = \sin(u(1)),$$

   a nonlinear point functional. Compute the directional derivative of f at $u(x)$ in direction $h(\cdot)$.
2. Let H be a real Hilbert space with scalar product $(\cdot, \cdot)_H$ and induced norm $\|\cdot\|_H$. Compute the Gâteaux derivative of

   $$f(u) = \|u\|_H^2.$$

3. Let V and W real Hilbert spaces and $u_d \in W$ be a fixed function. Let $S : V \to W$ be a linear and bounded operator. Compute the Fréchet derivative of

   $$F(u) = \|S(u) - u_d\|_W^2.$$

Exercise 3

[Optional]
Let $V = W = C[0,1]$ (thus the continuous functions on the closed interval $[0,1]$). We define the integral operator $F : V \to W$ as

$$F(u(x)) = \int_0^1 e^{x-s} u(s)\, ds, \quad x \in [0,1].$$

1. Show that F is linear.
2. Show that F is continuous.
3. Now let $V = W = L^2(0,1)$. Show that F is a continuous operator from V to W.
4. Slightly change the above operator to

   $$F(u(x)) = \int_0^x e^{x-s} u(s)\, ds, \quad x \in [0,1].$$

   Compute the adjoint operator F^*, i.e.,

   $$(w, Fv)_W = (F^* w, v)_V \quad \forall w \in W,\, v \in V,$$

   here $(\cdot,\cdot)$ denotes the L^2 scalar product.