Optimization

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1 Introduction

The goal of this short course (18 hours) is to recall the basics of optimization theory and algorithms. After recalling some definitions and notations, we shall focus entirely on continuous optimization. Several examples of optimization problems will be given as a motivation, arising from calculus of variations, control theory, inverse problems, image processing, optimal design and various applications fields. The case of finite dimensional optimization will be briefly discussed but the main emphasis, at least concerning existence of optimal solutions, will be on infinite dimensional optimization. Another part of the course will focus on numerical algorithms.

2 Compact outline and literature

- Class 1: Examples, problem statement, gradient descent, conjugate gradient, Newton
  Notes: handwritten on web page (Wick)

- Class 2: Existence of minimizers, lower semicontinuity, Ascoli-Arzelà, direct method in calculus of variations
  Notes: chapter 3 in lecture notes by Chambolle on web page.

- Class 3: Convex analysis, strong convexity in Hilbert spaces, projection on a closed convex set
  Notes: Grégoire Allaire [1, 2] (chapters on optimization, respectively).

- Class 4: Espaces de Hilbert, Convexité, Différentiabilité (Fréchet) Quelques résultats d'existence : projection sur un convexe, théorème de représentation de Riesz, minimisation des fonctions fortement convexes, minimisation des fonctions convexes. Conditions d'optimalité (dans un convexe).

- Class 5: Conditions d'optimalité, minimisation sous contrainte, contraintes égalité, extrema liés (multiplicateurs de Lagrange) contraintes inégalité, qualification, lemme de Farkas, points selles, dualité, théorème de Kuhn et Tucker
- Class 6: optimisation & algorithmes (par exemple gradient, Uzawa)

3 Introductory examples and algorithms

3.1 Problem statement

Let $X \subset R^N$ and $f : X \to R$ a function. The basic problem reads:

$$min f(x) \quad \text{s.t.} x \in X.$$

French: section 9.1 of [2]. English: section 9.1 of [1], section 1.1 of [8].

3.2 Examples and applications

French: section 9.1 of [2]. English: section 9.1 of [1], section 1.1 of [8].

3.3 Gradient descent for unconstrained optimization

Armijo step size, convergence, speed of convergence
English: chapter 2.2 of [8], chapter 3 and 5 of [7].

3.4 Newton for unconstrained optimization

Basics, convergence, globalization
English: chapter 2.2 of [8], chapter 3,6,7,11 of [7]. French: section 10.5 of [2].

4 Basic principles of functional analysis

4.1 Banach spaces

English: section 3.1 in [6].

4.2 Hilbert spaces

French: sections V.1 of [4]. English: sections 5.1 of [5], section 4.1 in [6].

4.3 Examples: Lebesgue L^p spaces

French: sections IV.1 and IV.2 of [4]. English: sections 4.1 and 4.2 of [5], section 3.4 in [6].

4.4 Duality

Linear and continuous applications, duality, bidual. Dual space of L^p.

English: section 3.5 in [6].

4.5 Weak convergence

Definitions of weak and weak-* sequential convergence.

French: section III.2 of [4]. English: section 3.2 of [5], section 5.12 in [6].

Definition of sequential compactness. Proof of sequential weak-* compactness (by a diagonal sequence extraction). Statement of sequential weak compactness. Weak lower semi-continuity of convex functionals.

French: sections III.4, III.5 and III.6 of [4]. English: sections 3.4, 3.5 and 3.6 of [5].

5 Convex analysis, existence of minimizers

5.1 Convex analysis

Convexity, strict convexity. Characterization with first and second-order derivatives.

Strong or $\alpha-$ convexity in Hilbert spaces.

Epigraph.

Projection on a closed convex set.

5.2 Lower semi-continuity

5.3 Existence results for convex minimization problems

Infinite at infinity condition. Coercivity.

Unconstrained and constrained minimization.

6 Optimality conditions

6.1 Unconstrained minimization

First-order and second-order conditions, sufficient and/or necessary conditions.

6.2 Minimization on a convex set

Euler inequality.

6.3 Equality and inequality constraints

Lagrange multipliers, qualification of the constraints, Farlas lemma, Karush-Kuhn-Tucker conditions, Kuhn-Tucker theorem, Lagrangian, dual problem, min-max problem.

7 Algorithms

7.1 Unconstrained minimization

Steppest descent or gradient algorithm, conjugate gradient, Newton algorithm.

7.2 Constrained minimization

Projected gradient algorithm, Uzawa algorithm, Arrow-Hurwitz algorithm, Newton algorithm.

Penalization of the constraints.

7.3 Sequential linear programming

Linear programming, dual problem, simplex algorithm, interior point methods.

Linear approximation.

7.4 Sequential quadratic programming

Minimization of a quadratic function with linear constraints: theory and algorithms.

Quadratic approximation.

References

[1] G. Allaire, *Numerical Analysis and Optimization. An Introduction to Mathematical Modelling and Numerical Simulation,* Oxford University Press, (2007).

[2] G. Allaire, *Analyse numériques et optimisation,* Éditions de l'École Polytechnique, 2ème édition, (2012).

[3] J. Bonnans, *Optimisation continue*, Mathématiques appliquées pour le Master / SMAI, Dunod, Paris (2006).

[4] H. Brézis *Analyse fonctionnelle*, Masson, Paris (1983).

[5] H. Brézis *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. Springer, New York (2011).

[6] Ph. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, SIAM, Philadelphia (2013).

[7] J. Nocedal, S. Wright, *Numerical optimization,* Second edition. Springer Series in Operations Research and Financial Engineering. Springer, New York (2006).

[8] M. Hinze, R. Pineau, M. Ulbrich, S. Ulbrich, *Optimization with PDE constraints*, Mathematical Modelling: Theory and Applications, Vol. 23, Springer, (2009).